MAXIMAL SMALL EXTENSIONS OF O-MINIMAL STRUCTURES

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ABSTRACT. A proper elementary extension of a model is called small if it realizes no new types over any finite set in the base model. We answer a question of Marker, and show that it is possible to have an o-minimal structure with a maximal small extension. Our construction yields such a structure for any cardinality. We show that in some cases, notably when the base structure is countable, the maximal small extension has maximal possible cardinality.

1. INTRODUCTION

Kueker, referenced in [HS91], defines:

Definition 1.1. Let $M \prec N$ be models. N is a *small extension of* M if, for any $a \in N$ and finite $A \subset M$, the type $\mathrm{tp}(a/A)$ is realized in M.

Kueker asked the question: for a general M, is there a "Hanf number" λ such that, if M has a small extension of any cardinality below λ, M has small extensions of every cardinality? Hrushovski and Shelah [HS91] answered this question for superstable M – in a countable theory it is $\beth_{\omega_1}$.

It is sometimes more convenient to work with a "maximal" small extension – a small extension that does not imply the existence of small extensions of every cardinality, and has maximal cardinality among small extensions like this. The existence of a maximal small extension gives the Hanf number as two more than the cardinality of this extension. Marker [Mar86] showed that any maximal small extension of an o-minimal structure could have cardinality at most $2^{|M|}$. Marker's argument uses the fact that there are at most $2^{|M|}$ types over M, so that an extension of greater cardinality would have to realize at least one type more than once. Since there are actually at most $\mathrm{Ded}(|M|)$ types over M, where

$$\mathrm{Ded}(\alpha) = \sup\{|\bar{Q}| : Q \text{ a linear order, } |Q| \leq \alpha\}$$

and $\bar{Q}$ denotes the completion of the linear order Q, Marker's argument shows that a maximal small extension must have cardinality at most $\mathrm{Ded}(|M|)$.

Recently, Kudaibergenov [Kud08] has shown that for weakly o-minimal theories with atomic models (a class that includes o-minimal theories), the above result can be tightened to say that a maximal small extension must have cardinality at most $2^{|T|}$, where $|T|$ is the cardinality of the theory.

Most examples of o-minimal structures either have no small extensions or unboundedly many – in a pure dense linear order, every extension is small. In the

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rationals as an ordered group, no extension is small. In fact, no non-trivial examples of o-minimal structures with both small and non-small extensions were known (obvious "glueing" examples can be constructed).

In this paper, we construct the first known example of an o-minimal structure with a maximal small extension. This construction generalizes to construct such examples for all cardinalities. In the countable case, our maximal small extension has cardinality $2^{\aleph_0}$, which is as large as possible. For some larger cardinalities, our maximal small extension will have cardinality equal to the corresponding Dedekind number. In general, though, we cannot show optimality, although for example the Generalized Continuum Hypothesis or a similar assumption would make our maximal small extensions have greatest possible cardinality.

The results in [Kud08] imply that if M is a structure with a maximal small extension and $|M| = \alpha$ for some cardinal α, then the cardinality of the theory T must be at least large enough that $2^{|T|} > \alpha$. Again, for each cardinal α our construction may give M with minimal $|T|$, subject to set-theoretic considerations.

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2. Types

Given $p, q \in S(M)$, say that q is *definable* from p if for any $c \models p$ there is $d \models q$ with $d \in \mathrm{dcl}(Mc)$.

Definition 2.1. Given a model, M, and a set, $A \subseteq M$, let a type $p \in S_1(M)$ be *A-finite* iff for some finite $\bar{b} \in A$, the type $p \restriction \bar{b}$ generates p. The type $p \in S(M)$ is *almost A-finite* iff there exists an A-finite type that is definable from p.

Since order-type implies type in o-minimal theories, A-finiteness has an interpretation in the order – $\mathrm{dcl}(\bar{b})$ is dense in M near c a realization of p, for $\bar{b} \subseteq A$ the witness to A-finiteness. Considering this interpretation, we have:

Remark 2.2. Let M be o-minimal and N an elementary extension of M. If N realizes no M-finite types then N is a small extension of M.

We recall here a classification of types given in [Mar86]:

Definition 2.3. A 1-type, $p \in S(A)$, $A = \mathrm{acl}(A)$, is *non-principal* iff p is not algebraic, p contains formulas of the form $x < a$ and $a < x$, and for each formula of the form $a < x$, there is $b \in A$ with $b < x$ in p, and similarly for $x < a$. p is *principal* iff it is not algebraic and not non-principal. A non-principal p is *uniquely realizable* if the prime model realizing p has just one realization of p.

3. Existence of Maximal Small Extensions

Proposition 3.1. *For every α, there is an o-minimal structure, M, $|M| = \alpha$, with small extensions but not unboundedly large small extensions. Moreover, if α is of the form $\beta^{<\lambda}$, for some λ, a small extension can be found of cardinality β^{λ}.*

Proof. We give a construction of models M and N, with $M \preceq N$, and N a maximal small extension of M. We then verify the sizes of M and N.

Let G be a divisible ordered abelian group, λ an ordinal, Q a dense divisible subgroup of G. Let $Q' = G \setminus Q$.

Let $M = G^{<\lambda}$. We consider M as a subgroup of G^λ, which is ordered lexicographically and equipped with group structure component-wise. Let our language be that of an ordered group, extended by constants for every element of $Q^{<\lambda}$. We will build N in stages.

- Let $M_0 = M$.
- Given M_i, choose $a \in G^\lambda$ such that any $b \in \mathrm{dcl}(aM_i) \setminus M_i$ has cofinal components in Q'. Let $M_{i+1} = \mathrm{Pr}(M_i a)$ (the prime model containing M_i and a). If no such a exists, then we halt.
- Take unions at limits.

This construction must halt at some point, since there are $\leq |G|^\lambda$ elements to add. Let the union of the M_i's be N.

M is o-minimal, since it is a divisible ordered abelian group, and each M_i and N is an elementary extension, since they are also divisible ordered abelian groups and this theory has quantifier elimination.

It remains to be shown that N is a small extension of M, and that there is no larger small extension of M. In fact, we show that every small extension of M comes from this type of construction.

Notation: we use M' to denote an arbitrary M_i or N. For $\alpha < \lambda$, $a[\alpha]$ is the α-th component of a, and $a \restriction \alpha = \langle a[i] \rangle_{i<\alpha}$. We let $a \sqsubset b$ denote "a is an initial segment of b."

Lemma 3.2. *Every principal type over M' is almost M-finite.*

Proof. Let p be principal over M'. Let p be generated by the formulas $\{a < x\} \cup \{x < e \mid e \in M', e > a\}$ (the other cases are similar). Let d be any realization of p. The type of $d - a$ over M' is generated by $\{0 < x\} \cup \{x < e \mid e > 0\}$ – the principal type near 0. Given any $e > 0 \in M'$, let α be the first index at which $e[\alpha] \neq 0$. Let $c \in Q^{\alpha+1}$ be such that $c[i] = 0$ for $i < \alpha$, and $0 < c[\alpha] < e[\alpha]$. Then $0 < c < e$. Thus, $x < c$ implies $x < e$, and $d - a < c$, so $\mathrm{tp}(d - a/M')$ is generated by $\mathrm{tp}(d - a)$.

□

Definition 3.3. Let $p \in S_1(M')$ be non-principal. p is *reducible* if there is $\alpha < \lambda$ such that, for any $a, b \in M'$, if $a \restriction \alpha = b \restriction \alpha$ then $x < a \in p \iff x < b \in p$.

Note that if p is reducible then it is not uniquely realizable, since for α as in the definition and any a with $\mathrm{length}(a) > \alpha$ and $a \restriction \alpha = 0$, if $c \models p$, then $c + a \models p$.

Lemma 3.4. *If $p \in S_1(M')$ is reducible, then p is M-finite.*

Proof. Let α be the least such in the definition of reducible. For each $\beta < \alpha$, we can find $a_\beta \in M$, a_β^+, a_β^- extending a_β such that $\mathrm{length}(a_\beta) = \beta$, $x < a_\beta^+ \in p$, and $x > a_\beta^- \in p$. It is easy to see that $\beta < \beta'$ implies $a_\beta \sqsubset a_{\beta'}$. Let $a = \bigcup_{\beta<\alpha} a_\beta$. Then $a \in M$.

Let d realize p, let e be any element of M'. WLOG, assume $e > d$. We show $e > d$ is implied by $\mathrm{tp}(d/a)$.

Case 1: $e \restriction \alpha \neq a \restriction \alpha$. Then e and a differ at some coordinate $\beta < \alpha$, so $e[\beta] > a[\beta]$, since $e > d > a \restriction \beta + 1$.

If $a > d$, we are done. Otherwise, by density of Q, we can find $c \in Q^{\beta+1}$ with $c[i] = 0$ for $i < \beta$, and $a[\beta] < a[\beta] + c[\beta] < e[\beta]$. Again, it is clear that $a + c > d$, so we are done for this case.

Case 2: $e \restriction \alpha = a \restriction \alpha$. Since we assume $e > d$, we also have $a > d$, since p is reducible. Let $\beta \geq \alpha$ be the first coordinate at or past α at which e is not 0 (if β does not exist then $e = a$). If $e[\beta] > 0$, we are done (since $e > a > d$), so let $e[\beta] < 0$. Choose $c \in Q^{\beta+1}$ such that $c[i] = 0$ for $i < \beta$, and $c[\beta] < e[\beta] < 0$. Then $c + a < e$, but since $(c + a) \restriction \alpha = e \restriction \alpha$, that means $c + a > d$, so $c + a > x \in \mathrm{tp}(d/a)$, and hence $e > d$ is implied by $\mathrm{tp}(d/a)$.

$\square$

Lemma 3.5. *If $p \in S_1(M')$ is non-reducible, then for some $a \in G^\lambda$, $\mathrm{tp}(a/M') = p$.*

Proof. For each $\alpha < \lambda$, by non-reducibility, there are $a^-_\alpha, a^+_\alpha \in M'$ such that $a^-_\alpha \restriction \alpha = a^+_\alpha \restriction \alpha$, but $a^-_\alpha < x < a^+_\alpha \in p$.

Let $a_\alpha = a^-_\alpha \restriction \alpha$. It is easy to check that $\alpha < \alpha'$ implies $a_\alpha \sqsubset a_{\alpha'}$.

Let $a = \bigcup_{\alpha<\lambda} a_\alpha$. If $a < e$, then at some component, say α, $a[\alpha] < e[\alpha]$. But $a \restriction \alpha + 1 = a^+_{\alpha+1}$, so $a^+_{\alpha+1} < e$, so $x < e \in p$.

The case $e < a$ is symmetric. Thus, $\mathrm{tp}(a/M') = p$.

$\square$

Given a sequence, a, we say that "a has cofinal components with property P" if for any $\lambda < \mathrm{length}(a)$, there is $\kappa > \lambda$ such that $a[\kappa]$ has property P.

Lemma 3.6. *Let $d \in G^\lambda$ realize a non-reducible type over M' without cofinal components in Q'. Then $\mathrm{tp}(d/M')$ is M-finite.*

Proof. For some $m < \lambda$, $b = d \restriction m$ has all the components of d in Q'. Note that $b \in M$. Given any $e \in M'$ with $x < e \in \mathrm{tp}(d/M')$, let n be the first index at which d and e differ.

If $n < m$, let $c \in Q^{n+1}$ be such that $c[i] = 0$ for $i < n$, and $0 < c[n] < e[n] - b[n]$. Then $x < b + c$ is in $\mathrm{tp}(d/b)$, and $b + c < e$.

If $n \geq m$, then choose $c \in Q^{n+1}$ such that $c[i] = 0$ for $i < m$, $c[i] = a[i]$ for $m \leq i < n$, and $d[n] < c[n] < e[n]$. Then $x < b + c$ is in $\mathrm{tp}(d/b)$ and $b + c < d$.

The $e < x$ case is symmetric.

$\square$

Lemma 3.7. *If $d \in G^\lambda \setminus M'$ has cofinal components in Q' , then $\mathrm{tp}(d/M')$ is not M-finite. Thus, if every $b \in \mathrm{dcl}(dM') \setminus M'$ has cofinal components in Q', then d is not almost M-finite.*

Proof. Assume for a contradiction that $\mathrm{tp}(d/M')$ is M-finite. Let $\bar{b} = (b_1, \dots, b_m) \in M^m$ witness this, of minimal length (as a tuple).

For any $a \in M'$, we can find $f(\bar{b})$, with f $\emptyset$-definable, such that $f(\bar{b})$ lies between d and a. Considering $d \restriction i$, for $i < \lambda$, we can find $\{f_i(\bar{b})\}_{i<\lambda}$ with $f_i(\bar{b}) \restriction i = d \restriction i$.

By quantifier elimination for divisible ordered abelian groups, we know that each $f_i(\bar{b})$ is an affine linear combination (with rational coefficients) of the b_j's, with the affine part given by $c \in Q^{<\lambda}$. If we take $\alpha = \max(\mathrm{length}(b_j) \mid j \leq m)$, then for any β, $f_\beta(\bar{b})$ can have no components in Q' past the αth one. But this is clearly impossible.

$\square$

This completes our proof that N is a maximal small extension of M. N is certainly a proper extension of M, since any element with cofinal components in Q' can be adjoined to form M_1. It remains to determine its size. We lose nothing

by restricting to the case where λ is an infinite cardinal. We consider G^λ and its divisible subgroups as $\mathbb{Q}$-vector spaces.

Claim 3.8. *Let β be a cardinal such that there exist linearly independent $\{a_i\}_{i<\beta} \in G^\lambda$ with $(M + Q^\lambda) \oplus \operatorname{span}(\{a_i\}_{i<\beta}) = G^\lambda$. Then in the construction above we can ensure that $|N| = \beta + |M|$.*

Proof. We show the claim by showing that we can take a_i to be the element adjoined to M_i to produce M_{i+1}, for every $i < \beta$. For any such i, the element a_i has the property that every element of $\operatorname{dcl}(Ma_{\leq i}) \setminus \operatorname{dcl}(Ma_{<i})$ has cofinal components in Q', since otherwise that element would be in the span of $M + Q^\lambda + \operatorname{span}(\{a_j\}_{j<i})$, and $\operatorname{span}(\{a_j\}_{j<\beta})$ is linearly independent from $M + Q^\lambda$. □

Let W be a divisible subgroup of G such that $G = Q \oplus W$. Then $G^\lambda = Q^\lambda \oplus W^\lambda$. Let $\gamma = |W|$.

Claim 3.9. $\dim W^\lambda = \gamma^\lambda$.

Proof. Note that $|W^\lambda| \geq \gamma^\lambda \geq 2^{\aleph_0}$, since any element can be uniquely written as a λ-sequence of elements of W. Since we are considering W^λ as a vector space over $\mathbb{Q}$, a countable field, it follows that $\dim(W^\lambda) = |W^\lambda|$. □

We can write $W^\lambda = W^{<\lambda} \oplus X$, for some divisible subgroup X of W^λ.

Claim 3.10. $(M + Q^\lambda) \oplus X = G^\lambda$.

Proof.

$$\begin{aligned} G^\lambda &= Q^\lambda \oplus W^\lambda = (Q^\lambda \oplus W^{<\lambda}) \oplus X \\ &= ((Q^{<\lambda} + W^{<\lambda}) + Q^\lambda) \oplus X = (G^{<\lambda} + Q^\lambda) \oplus X = (M + Q^\lambda) \oplus X. \end{aligned}$$

□

This implies that we may let the desired sequence $\{a_i\}_{i<\beta}$ be given by a basis for X.

Claim 3.11. $\dim X = \gamma^\lambda$.

Proof. We construct a set of independent (even over $W^{<\lambda}$) elements of W^λ, with size γ^λ and each element of length λ, showing that $\dim X \geq \gamma^\lambda$, which is enough.

Since $\lambda \times \lambda = \lambda$, we can find λ disjoint subsets of λ of length λ (necessarily cofinal). Let $\{X_i \mid i < \lambda\}$ be the characteristic functions of these subsets – each X_i is a binary sequence of length λ. For $b \in W$, let bX_i denote the element of W^λ obtained by replacing each 1 in the sequence X_i by b.

For $f \in W^\lambda$, let $A_f = \sum_{i<\lambda} f(i)X_i$. This sum is well-defined, because no two X_is are non-zero on the same component. We know that there is a basis of W^λ of size γ^λ, say $\{f_j\}_{j<\gamma^\lambda}$. Denote A_{f_j} by A_j. We show that $\{A_j \mid j < \gamma^\lambda\}$ is linearly independent and its span is disjoint from $W^{<\lambda} \setminus \{0\}$. Without loss of generality, it is enough to show that no non-zero linear combination of $A_1, \ldots, A_n$ is in $W^{<\lambda}$.

Suppose that $q_1A_1 + \ldots + q_nA_n = c$, where $q_j \in \mathbb{Q}$, $c \in W^{<\lambda}$. This then implies that $\sum_{i<\lambda}(\sum_{j\leq n} q_j f_j(i))X_i = c$. Fix $i < \lambda$. If $k, l \in X_i$, then it is clear that the left-hand side has the same value at its k and l components. But if we choose $k < \operatorname{length}(c) < l$, then the lth component must be 0, so the kth component is too. Since this holds for every $i < \lambda$ and $k < \operatorname{length}(c)$, this implies $c = 0$. But

this implies $\sum_{j\leq n} q_j f_j = 0$, and so $q_j = 0$, $j \leq n$, and hence the A_is are linearly independent.

□

Claim 3.8 applied to a basis of X, using Claims 3.10 and 3.11, shows that $|N| = |M| + \gamma^\lambda$, where γ is the cardinality of W, a divisible subgroup of G with $Q \oplus W = G$. The next section shows how this implies the cardinality statements of the proposition. □

4. Cardinalities

When $\lambda = \aleph_0$, G is the real closure of $\mathbb{Q}$, and $Q = \mathbb{Q}$, then $|M| = \aleph_0$ and $|N| = 2^{\aleph_0}$: the bound is as sharp as possible. In general, for any α an elementary compactness argument shows there exist G and Q such that $|G| = |W| = \alpha$. If we take λ to be $\aleph_0$, then $|M| = \alpha$. However, while N exists, it is possible that $|N| = |M|$, since $\alpha^{\aleph_0}$ may be α. Note, though, that N is still a proper maximal small extension of M, examples of which were not known before.

M can have any cardinality of the form $\alpha^{<\lambda}$ for any two infinite cardinals α, λ. Then N can have cardinality at least α^λ. This corresponds to the full tree of height λ with α-many branchings at each node. Note that the definition of $\mathrm{Ded}(-)$ can be rephrased in terms of trees, so that in fact $\mathrm{Ded}(\beta)$ is the sup of the cardinalities of the completions of trees of cardinality β. It is clear that if a cardinal β is of the form $\alpha^{<\lambda}$, then $\mathrm{Ded}(\beta) = \alpha^\lambda$. Thus, for such cardinals β, this construction shows that the maximal small extension has cardinality $\mathrm{Ded}(\beta)$.

In general, it is not hard to see that the construction above can be done with G^λ replaced by $\prod_{i<\lambda} G_i$, where each G_i is a divisible ordered abelian group, Q^λ replaced by $\prod_{i<\lambda} Q_i$, where each Q_i is a dense divisible subgroup of G_i, and M the subgroup consisting of all elements of N of length $< \lambda$. It is an open question, potentially independent from ZFC, whether such a configuration can actually witness the Dedekind number for every cardinal. It would not if there were some cardinal, α, with $\alpha \neq |M|$ for any such M, but $\mathrm{Ded}(\alpha) > |N|$ for any corresponding N. That would require that $\mathrm{Ded}(\alpha)$ be witnessed by a highly asymmetric tree, but no results are known on the types of trees that are needed to witness Ded.

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